Recall: A set A has

finite: • size n if there's a bijection $(n \in \mathbb{N})$

$\{1, \ldots, n\} \longrightarrow A$

denumerable: • size $\aleph_0$ ————————

$\{1, 2, \ldots\} \longrightarrow A$

If A is either finite or denumerable, A is countable.

We proved • preserved by $\cup$, $\cap$, $\times$.

$\mathbb{N}$ denumerable (definition)

$\mathbb{Z}$ denumerable, $\mathbb{Q}$ denumerable.

Theorem (Cantor): $\mathbb{R}$ is not countable

Proof will use "Cantor diagonalization".

Note denumerable $\equiv$ $\exists$ bijection $\mathbb{N} \longrightarrow A$

countable $\equiv$ $\exists$ surjection $\mathbb{N} \longrightarrow A$

uncountable $\equiv$ $\nexists$ surjection $\mathbb{N} \longrightarrow A$

$\equiv \forall f: \mathbb{N} \to A$, f is not a surjection.

Kronecker: "God made the natural numbers, everything else is the work of man."

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ used for counting.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ "I want to subtract"

$\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\}$ "I want to divide"

$\mathbb{R} = \longleftrightarrow$ "I want to take <u>limits</u> of converging sequences."

<u>Decimal representation:</u>

<u>Def:</u> A <u>decimal representation</u> is an infinite sequence of digits

$\underbrace{0\ 0\ 0}\ .\ d_1\ d_2\ d_3\ d_4 \ldots$ with each $d_i \in \{0, 1, \ldots, 9\}$.

ie, decimal representations of $0 \le x \le 1$

The limit of

$\frac{d_1}{10}, \frac{\overline{d_1 d_2}}{100}, \frac{\overline{d_1 d_2 d_3}}{1000}, \ldots$ is
- $\to$ rational, if the sequence is eventually repeating
- $\to$ irrational if not.

Ex: $\frac{3}{5} = 0.600000\ldots$

$$\frac{47}{200} = \frac{235}{1000} = .2350000\ldots$$

- If a decimal representation ends in an infinite tail of zeroes, it is of the form $\frac{\#}{10^n}$.

Ex: $\frac{1}{3} = .33333\ldots$

The reason is because

$$\frac{3}{10},\ \frac{33}{100},\ \frac{333}{1000}\ \ldots \text{ converges to } \frac{1}{3}.$$

$$.3,\ .33,\ .333,\ \ldots$$

Ex: $\pi = 3.1415926535\ldots$ is the limit of a sequence

$$3,\ \frac{31}{10},\ \frac{314}{100},\ \frac{3141}{1000},\ \frac{31415}{10000},\ \ldots$$

Ex: $\frac{1}{11} = .09090909\ldots$

(*)
Slight wrinkle: If a rational number has a finite decimal representation, then it has two possible representations.

Ex: $.0236 = .02359999\ldots$

Proof of Theorem: Let $f : \mathbb{N} \longrightarrow [0,1]$ be any function. I will prove f is not surjective.

For each $n \in \mathbb{N}$, write $f(n)$'s decimal representation.

$$f(n) = .d_{n,1}\, d_{n,2}\, d_{n,3} \cdots \quad \text{where } d_{n,k} = k\text{-th digit of } f(n) \text{ after the decimal point.}$$

$$f(1) = .\boxed{d_{1,1}}\, d_{1,2}\, d_{1,3}\, d_{1,4} \cdots$$
$$f(2) = .d_{2,1}\, \boxed{d_{2,2}}\, d_{2,3}\, d_{2,4} \cdots$$
$$f(3) = .d_{3,1}\, d_{3,2}\, \boxed{d_{3,3}}\, d_{3,4} \cdots$$
$$f(4) = .d_{4,1}\, d_{4,2}\, d_{4,3}\, \boxed{d_{4,4}} \cdots$$
$$\vdots \qquad \vdots \qquad \ddots$$

Construct a new number

$$x = .e_1 e_2 e_3 e_4 \ldots$$

according to the rule:

If $d_{n,n} = 0$,	then $e_n = 4$
— 1	— 4
— 2	— 4
— 3	— 4
— 4	— 3
— 5	— 4
— 6	— 4
— 7	— 4
— 8	— 4
— 9	— 4

The choice made at right was arbitrary and is just to ensure we never end up with x being a "finite decimal representation."

Note that $\forall n \in \mathbb{N}$,
n-th digit of x $\neq$ n-th digit of $f(n)$
and therefore, $x \neq f(n)$ (Because x is not one of the numbers (*) with a finite representation.)

So it follows f is not surjective. ∎